Illuminating the Poincare Recurrence Theorem with Information Technology

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Abstract-**In the beginning of the Twentieth Century Poincare recurrence theorem revolutionized modern theory of dynamical systems and statistical mechanics. Indeed, the problem of recurrence times lies in the very essence of discrete mathematics and statistical mechanics. The meaning of the theorem is that distant parts of the phase space repeatedly visited by the trajectory of the dynamical system possessing some ergodic properties, an argument used by Zermelo against Boltzmann. For very small cells of the phase space the time needed for such a revisiting becomes astronomically large. A suspicious reader could assume that this is not true for usual partitions. Indeed, in a highly specialized series of papers, Balakrishnan, and Nicolis addressed the problem of recurrence in a firm basis, introducing definitions, terminology, theorems and numerical experimentation. However, at least these computational investigations have been restricted to the coarse-grained views of the theorem, what we call here the "global views". Motivated by this call, we here illuminate the many facets of Poincare recurrence time theorem for an archetype of a Complex System, the logistic map. We do not have restricted ourselves to the global view of the things, but we occasionally escape towards a more "local" view. In the present work we deliberately address both views in some detail. In particular information technology obeying Moore's law allows to everybody to perform careful numerical experiments for the study of statistical mechanics. According to the philosophy of Experimental (or Organic) Mathematics, the idea that "real mathematicians do not compute!" is seriously put in question. Modern information technology much like a telescope sheds light also in the theory of dynamical systems.**

Keywords-Dynamical systems, Poincare recurrence theorem, recurrence times, logistic map, ergodicity

I. INTRODUCTION

In the beginning of the *20th* century, Poincare recurrence time theorem has revolutionized modern Mechanics and the foundations of Statistical Physics ([1],[2]).However, most of the first considerations have been focused in the case where the diameter of the cell is very small, so that the recurrence time becomes astronomically large. This fact, gives the feeling that this is true for the Poincare times in general.

Most of the specialized recent papers on the subject have as a first scope to bypass this difficulty: when the diameter of the cell in the phase space assumes ordinary values, the recurrence times are also non-divergent numbers ([14], [15], and [16]).

In view of its profound character and importance, a highly specialized work on the recurrence times of discrete-time maps has been completed by V. Balakrishnan, G. Nicolis and C. Nicolis ([11]). In particular, the authors addressed the recurrence time statistics and related properties of chaotic, dissipative, noninvertible maps. Careful numerical investigations have also been exhibited. A continuation of this study is the extension of the theory to continuous time dynamical systems by the same authors ([15]).

In previous work we have restricted ourselves to a "version" of Poincare theorem tailored for symbolic dynamics and especially for the symbolic dynamics of the generating R-L partition of the logistic map ([23]). It should be very interesting in the immediate future to "escape" from this "static" and "un-hydrated" picture of the Poincare recurrence time theorem and pass to a dynamic investigation changing also the initial points and diameters of the cells, point by point. Other directions of research could also being envisaged ([22]).

In the present paper we try a comprehensive global-versus-local study of the recurrence times for the logistic map. Although many scattered results are well known, a detailed study of the global-versus-local issue lacks from the literature. This work tries to fill the gap.

Furthermore, the mean recurrence time as a function of diameter of the different cells considered here is for different control parameter values r.

The paper is articulated as follows. In Sec II, we introduce the general formulation of the Poincare recurrence theorem. A description is given in terms of the Frobenious–Perron equation and the problem of recurrence is rigorously introduced. In Sec. III, the main archetype of a complex system, the logistic map, is introduced in detail. The connection between complex behavior and classical chaos is outlined. For reasons of clarity, rigorous as well as effective definitions of the maximum first return time and the mean recurrence time are explicitly given. In Sec IV we speak about our numerical experimentation and a connection with the notion of ergodicity theorem is established. In Sec V numerical results and the corresponding Figures are introduced .In Sec VI we draw the basic conclusions and discuss future research plans.

II. FORMULATION

Recurrence time statistics provides information on the nature of the processes going on in a dynamical system in a variety of contexts, ranging from the foundations of statistical mechanics ([1]-[5],[7],[21],[23]) to the classification of atmospheric "analogs" and the prediction of short-term weather fluctuations ([6]).

In its classical version, Poincare recurrence theorem refers to a one-parameter family F'of one-to-one measure-preserving transformations. It states that if C is a subset of the phase

space Γ such that $\mu(C)>0$ [μ is a completely additive measure with $\mu(\Gamma)=1$], then for almost every point $P\in C$ there exists a sufficiently large t such that $F^{t}P\in C$. By discretizing time in slices of duration τ and further assuming that the transformation F^{t} is metrically transitive (ergodic), one can then derive the following expression for the mean recurrence time,

$$<\theta_{\tau}> = \tau/\mu(C) \tag{1}$$

For reasons of completeness, and for further use, we compile here some important points of the general formalism for the problem of recurrence times, introduced in ([14]). Consider the discrete-time dynamical system

$$X_{n+1} = f(X_n, \mu) \quad n=0,1 \tag{2}$$

where f is the evolution law and μ the control parameter. It wi ll be assumed that under f the state variables X remain confine d to a finite, invariant part Γ of phase space. In what follows we shall be interested in evolution laws and parameter ranges for which the dynamics is chaotic in most cases. As is well kn own generates an evolution equation for the probability densit y $\rho_n(X)$, $X\in\Gamma$,namely, the Frobenious-Perron equation

$$\rho_{n+1}(X) = \int_{\Gamma} dY\, \delta(X- f(Y,\mu))\, \rho_n(Y) \tag{3}$$

To formulate the problem of recurrence, we consider a finite cell $C\in\Gamma$ and assume that the evolution is started at a point $x_0\in C$. As the evolution proceeds, the representative point will in general escape from C, but, unless it is a part of an exceptional (e.g. periodic) orbit or the system has poor ergodic properties, it will be reinjected repeatedly into *C*. Let

$$F(C,0;C,n)=\text{Prob}(X_0\in C, X_1\notin C,\dots,X_{n-1}\notin C, X_n\in C) \tag{4}$$

be the normalized probability of the first return of the representative point to the cell Cat time n.

The mean recurrence time is then

$$<n_{cc}>=\sum_{n=1}^{\infty} nF(C,0;C,n) \tag{5}$$

Higher order moments of the recurrence time distribution can be defined similarly.

Since the deterministic evolution law implies that the members of an initial Gibbs ensemble are propagated in time by a δ-function type transition probability, F(C, 0; C, n) can also be expressed as

$$F(C,0;C,n) = \int_C dX_0\rho(X_0)\int_{\bar{C}} dX_1 \ldots\ldots \int_C dX_{n-1}\int_C dX_n \frac{\prod_{r=1}^{\infty} \delta(X_r - f^{(r)}(X_0,\mu))}{\int_C dX_0\rho(X_0)} \tag{6}$$

Where ρ is the invariant density and $\bar{C}$ is the complement of C in Γ. In practice, the explicit evaluation may not be feasible. For this reason we resort to the statistical description afforded by the Frobenious-Perron equation.

However, this is not directly applicable for arbitrary maps, as is for instance the case for the following Section.

III. LOGISTIC MAP

Let us elaborate. We first introduce the logistic map in its familiar form ([3]-[5], [7], [13], [16])

$$x_{n+1} = rx_n(1-x_n), \tag{7}$$

For the logistic map the generating partition is well-known, following an argument dating back to the French Mathematician Gaston Julia ([11]). To be more specific, the partition of the phase space (which in this case is the unit interval I=[0,1]) L=[0,0.5] and R=[0.5,1] is a generating partition. In this manner the trajectory of the system in the continuous unit time interval is projected with a one-to-one topological correspondence to a coarse-grained symbolic trajectory. The information content of the symbolic trajectory is the "minimum distinguishing information" in the words of Metropolis et al. ([12]). Needless to say, in this representation the logistic map is viewed as an abstract information generator ([16]-[21]).

Examining the generating partition, one introduces the Lyapunov exponent as a measure of unpredictable behavior. Roughly, when the Lyapunov is negative the system is periodic and when is positive chaotic.

Examining now not only the generating partition but arbitrary partitions in the phase space, one can measure the ergodic properties of the given partition.

To begin with, an important relevant quantity to define is the notion of the first return time. This is the number of iterations, in discrete time steps, which is needed for a point inside the cell to abandon the cell and to return for the first time N_{first} in the cell [see also the corresponding definitions, Sec II].

Let us elaborate. In a first step we consider two equal-sized typical, and non-overlapping cells in the phase space of the logistic map and we record the first return times N_{first} , as a function of the control parameter values r, where $3.55 < r < 4.00$. The starting points belong to the partitions. We have fixed an N_{max}=300,000.

For a given cell of the phase space, we compute about twenty control parameters from which we present in the Tables, r=FP (for the definition, see below), r=3.57, r = 3.70, 3.75, 3.8282, 3.8284, 3.90, 3.95, 4.00, and for two fixed initial conditions $X_0 = 0.3$ and $X_0 = 0.7$. We repeat this process for a spectrum of concentric cells in the phase space which share the same center X_0, but have diminishing diameters d_i (see Tables 1,2 and the Discussion of Section IV).

In the sequel, we explain the reasons justifying this choice of control parameter values.

For r=FP (FP=Feigenbaum Point =3.56994567…) the system is in the transition between stable periodic orbits and chaotic behavior ("at the edge of disorder") ([16]) .The system presents the non-chaotic multifractal attractor with a Lyapunov exponent which strictly vanishes (is strictly zero). Many theoretical and numerical works have as the basic subject this point, and analytical expressions are available for the mean value, the block entropies etc. These results allow for comparison with numerical studies and Experimental Mathematics.

For r=3.57 the system lies in the region of "weak chaos with a non-zero memory." The system is in a little (literary

epsilon) distance from the Feigenbaum point. The system is not mixing and obeys scaling laws.

For r=3.8282 and r=3.8284 we are inside the window of "parametric chaotic intermittency." The trajectory of the system is partially trapped by the period-3 orbit and the symbolic dynamics is a blend of periodic and chaotic motion with no end. The presence of ever-lasting transients is probable.

For r=3.95 the system is in the region of "developed chaos". The Lyapunov exponent of the trajectory assumes a high positive value (but not the maximum value) and the motion is irregular and erratic. However, we are not yet completely uncorrelated and some degree of correlation persists. Syntactic rules for the symbolic dynamics also persist.

For r=4.0 we are in the regime of "fully developed chaos". The motion is completely irregular, erratic and unpredictable. The symbolic dynamics of the system is equivalent to a "perfect coin tossing", that is a Markov partition. Analytic results for observables in this case are readily available under a closed form.

After this small digression and for reasons of clarity, we give the following definitions.

A. Definition of the Maximum First Return Time

Inside a given cell of diameter d_i, and keeping constant the control parameter value r, we consider a sequence of initial points $\{x_{i,}\}$,each one of them having his own first return time. Consequently, one has a correspondence between the sets $\{x_{i,}\}$ and $\{N_1;i\}$ and then, the maximum first return time is the maximum value of the set $\{N_1;i\}$

B. Definition of the Mean Recurrence Time

Inside a given cell of diameter d_i and keeping constant the control parameter value r, starting from one initial pointx_0, after a fixed number of iterations [see also Sec II, eq.(4)], one obtains a first return time N_{first}. Iterating from this point the map, we obtain a second return time N_2, etc. In this manner, for every given ordered set (d_i, x_0, r_i) corresponds a sequence of times N_{first} ,N_2,N_3, ,… . We are thus led to define, for instance, the mean recurrence time when the first 200,000 recurrences are taken over as:

$$\langle n\rangle = (N_{first}+N_2 +N_3 +\ldots+N_{200000}) / (200,000) \qquad (8)$$

See also eq. (5).

IV. .NUMERICAL EXPERIMENTATION

What do we expect from such numerical experiments?

1. To start with these computations, we perform two major numerical experiments for distinct partitions of the logistic map. Our basic task is to measure the first return time of the map. We run the logistic map, for two initial points x_0= 0.3 and x_0= 0.7. We have fixed a N_{uppe}=300,000 .We progressively reduce the cell diameter from 0.2 to 0.00625 and we change the control parameter value from r=FP to r=4.00 (see Tables 1, 2). Every time we keep the minimum number of iterations needed for the first return to the cell.

For instance, for the initial value x_0= 0.3 of the cell [0.2,0.4] and r=3.95 we find N_{first}=4, while for the initial value x_0= 0.7 of the cell [0.6,0.8] and r=3.95 we find N_{first} =15. Although there are important fluctuations, common is Statistical Physics, there are also similarities in the behaviors.

A very important remark, is that in Tables 1,2, recording the first return time as a function of the control parameter value for smaller and smaller diameters of the cells, the first return times became bigger and bigger for smaller and smaller diameters. For tiny partitions the first return time become astronomically large. In this way we find again the initial interpretation of the Poincare recurrence theorem.

2. In Tables 3, 4 and Figures 1, 2, 3 we present the maximum first return time as a function of the starting point. We find instructive to describe here the whole numerical experiment in detail.

We first consider three different cells in the phase space of the logistic map, namely [0.2, 0.4], [0.25, 0.35], [0.275, 0.325]. We fix the control parameter value at r=3.70, and we start iterating the map from three different X_0's, in order to take a (partial) clue of the dependence on the initial point. We next continue for other cells and initial points, taking in consideration the final points (denoted by X_{return}). As we know from the theory of dynamical systems, for a chaotic system the X_{return}may be very different.

From the Tables and corresponding Figures mentioned above, we conclude that the maximum first return time depends strongly on the initial point of the trajectory and the control parameter value of the map.

3. In the sequel, Tables 5-8, we present our third class of numerical experiments, which refer to the notion of the mean recurrence time defined rigorously in Sec III (Def. 2). As the mean recurrence time is arguably the principal object of our research plan here, we give it most space and detail.

We here explore an effective definition of the mean recurrence time, as computed by the statistical treatment of the iterations of the logistic map. Concretely, we obtain for two different initial conditions X_0=0.3 and X_0=0.7 the recurrence times of two families of cells. Every family is constituted by 11 cells whose successive diameter is divided by a factor 2. More specifically, considering initially the cell [0.2, 0.4] whose diameter is 0.2, we arrive to the 11 cell [0.2990234375, 0.30009765625] whose diameter is 0.0001953125. Similarly, we construct the second family of cells, using initially the cell [0.6,0.8] and the central point X_0=0.7.

In any case there is a matter of numerical precision (good statistics).

In order to confirm our statistical precision and verify general trends, we repeat the calculation for the first seven cells of the first 500,000 recurrence times. The results are confirmed up to one significant digit. These validate both the method and the results.

From Tables 5 to 8 we observe that for different r's, there are significant differences in the values of the mean recurrence time. However, there are also some general trends. So , what should remark is that as the diameter of the cell is reduced the mean recurrence time augments .There is only one exception to this rule , in Table 6 for r=3.70 for the cells [0.6 - 0.8] and [0.65 – 0.75] where the mean recurrence time turns from 5.9319 to 5.4889 .

We also point out that for all cells after diameter higher than 0.025 the mean recurrence time is doubled from cell to cell. (We shall come back to this matter in the next section.)

We proceed to depict (Figs. 4 -11) the mean recurrence times as a function of the control parameters , for cells of the same diameter and different initial points X_0 =0.3 and X_0=0.7 . The diameter is divided by a factor 2 .We note that for the preparation of the figures we consider more than exhibited in the Tables (we open the diagrams with r= 3.55). In particular for the following pairs:

[0.2; 0.4] and [0.6; 0.8] -> Fig.4

[0.25; 0.35] and [0.65; 0.75] -> Fig.5

[0.275; 0.325] and [0.675; 0.725] -> Fig.6 etc.

Let us summarize the basic conclusions from the inspection of the Fig.4- 11. In all pairs of curves in the diagrams there is a good accordance of the mean recurrence times. However, as we move towards partitions of smaller and smaller diameters the differences between the recurrence times of the cells become more and more important. To be concrete, in Fig .4 the fluctuations are negligible, but from Figs. 5–7 are amplified and omnipresent.

We now turn to the problem of the behavior of the mean recurrence time as a function of the diameter of the cells for r=3.70, r= 3.8282, r=3.95, r=4.00. We depict in Fig. 17 – 20 the dependence of mean recurrence time as a function of d_i (for d_i ranging from 0.2 – 0.0015625). The main observation here is that for any value of r, as $d_i \to 0$, $<n> \to \infty$.

One crucial remark is that for d_i very small, we obtain a power law in the diagram $<n> = f(d)$. In order to obtain the exponent of the power law, we depict the relation $<n> = f(d)$ in log – log plot, （in Fig. 21 – 24）. From the figures it is clear that $<n> \sim d^{-a}$, where “a” is the exponent.

Taking the logarithms of both parts we obtain $\log<n> = -a\log(d) + c$,where c is a constant . Applying the best fit method of least squares we obtain Tables 9 and 10. We obtain an excellent auto-regression coefficients for $d_i > 0.025$, for r=3.70, r=3.8282, r=3.95, r=4.00. Our main finding is that α = -1 within a very good precision.

This postulates the relation $<n> \sim \frac{1}{d}$, irrespectively of the cell and the control parameter value.

4.In the Figs. 13-16 we present a comparison of the maximum first return time $N_1$to the mean recurrence times with initial conditions X_0=0.3 , as a function of the control parameter value for $3.55 < r < 4.00$.

We observe a close accordance of the two curves. Furthermore when we trace the same diagram for the cell [0.675, 0.725] we notice similar trends.

Is this connected with the ergodic hypothesis?

What we know from the general theory of dynamical systems, is that dissipative systems are ergodic on the attractor.

The ergodic hypothesis in our case means that starting from any initial point (with the exception of a set of measure zero, with poor ergodic properties), the system converges to the attractor. Moreover this means that starting from an arbitrary initial point belonging to the set of good ergodic properties, we measure the same statistical values when following the trajectory. This exactly means that the mean value of any measurable quantity in time equals the mean value in space.

This proposition is known by a name proposed by Boltzmann: the ergodic theorem. Intuitively this proposition expresses the fact that almost every trajectory of the system spends equal times in equal regions everywhere in phase space.

This is manifested in our case by Fig. 4 – 12. In those Figures the notion of mean recurrence times is well defined and measured. Furthermore in Fig. 13 – 16 it is shown that the maximum first return time is almost equal to the mean recurrence time for d > 0.05.

We also notice the presence of the important statistical fluctuations. So, under certain conditions, the ergodic theorem tells us that the time average is equal to the space average.

V. NUMERICAL RESULTS

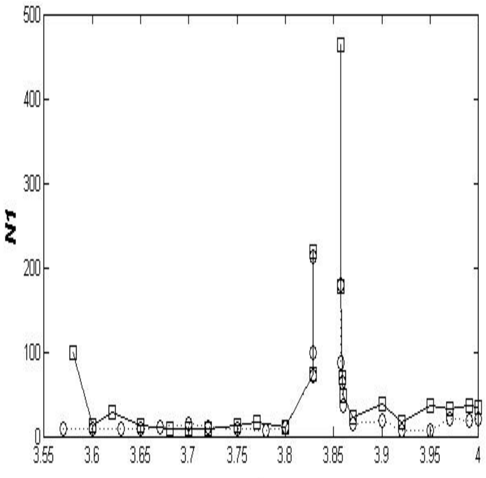

Fig.1.Maximum first return time $N_1$for the cells [0.2 , 0.4] and [0.6 , 0.8] of the logistic map as a function of the control parameter values r, where $3.55 < r < 4.00$. The starting points belong to the partitions. We have fixed an Nmax=300,000 The straight line stands for the second cell and the dashed line is for the first cell.

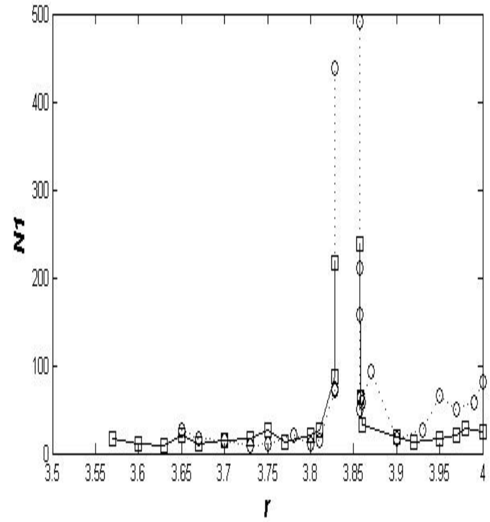

Fig.2.Maximum first return time N_1 for the cells [0.25 , 0.35] and [0.65 , 0.75] of the logistic map as a function of the control parameter values r, where $3.55 < r < 4.00$. The starting points belong to the partitions. We have fixed a Nmax=300,000. The straight line stands for the second cell and the dashed line is for the first cell.

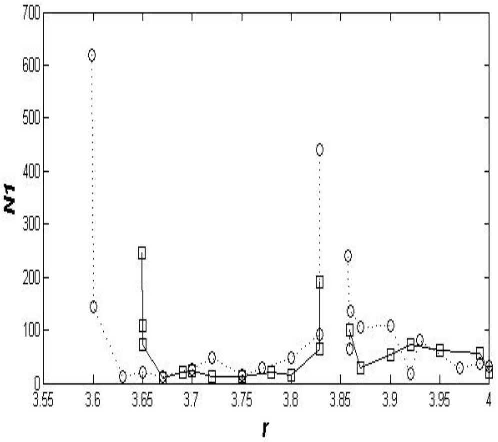

Fig.3.Maximum first return time N_1 for the cells [0.275, 0.325] and [0.675, 0.725] of the logistic map as a function of the control parameter values r, where $3.55 < r < 4.00$. The starting points belong to the partitions. We have fixed a Nmax=300,000. The straight line stands for the second cell and the dashed line is for the first cell.

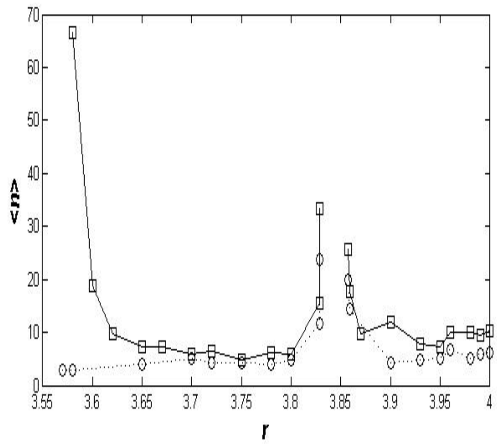

Fig.4. Mean recurrence time (<n>) for the cells [0.2, 0.4] and [0.6, 0.8], serves as a function of the control parameter value r, $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. We observe a good agreement of the two curves after r=3.65. The initial condition for the first cell is x_0= 0.3 and for the second cell is x_0= 0.7. The straight line stands for the second cell and the dashed line is for the first cell.

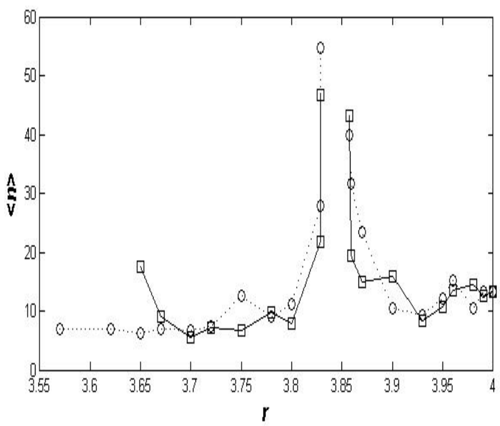

Fig.5. Mean recurrence time (<n>) for the cells [0.25, 0.35] and [0.65, 0.75], as a function of the control parameter value r , $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell is x_0= 0.3 and for the second cell is x_0= 0.7. The straight line stands for the second cell and the dashed line is for the first cell.

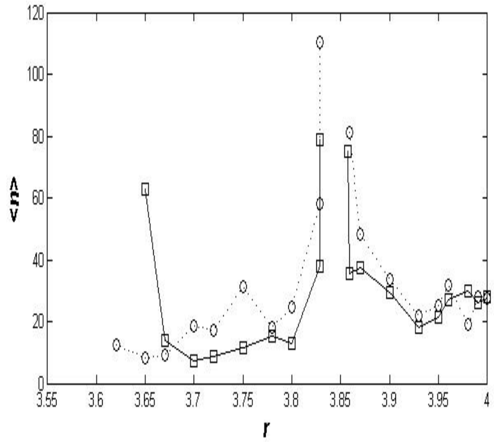

Fig.6.Mean recurrence time (<n>) for the cells [0.275, 0.325] and [0.675, 0.725, as a function of the control parameter value r, $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell is $x_0 = 0.3$ and for the second cell is $x_0 = 0.7$. The straight line stands for the second cell and the dashed line is for the first cell.

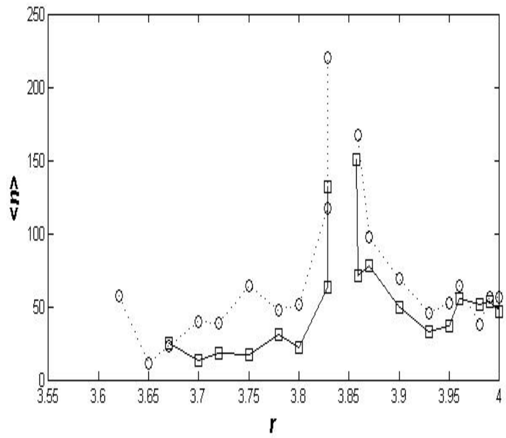

Fig.7. Mean recurrence time (<n>) for the cells [0.2875, 0.3125] and [0.6875, 0.7125], serve as a function of the control parameter value r, $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell is x_0= 0.3 and for the second cell is x_0= 0.7. The straight line stands for the second cell and the dashed line is for the first cell.

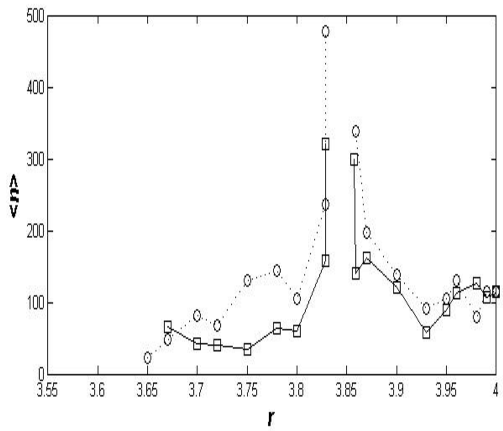

Fig.8. Mean recurrence time (<n>) for the cells [0.29375, 0.30625] and [0.69375, 0.70625], as a function of the control parameter value r , $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell is x_0= 0.3 and for the second cell is x_0= 0.7. The straight line stands for the second cell and the dashed line is for the first cell.

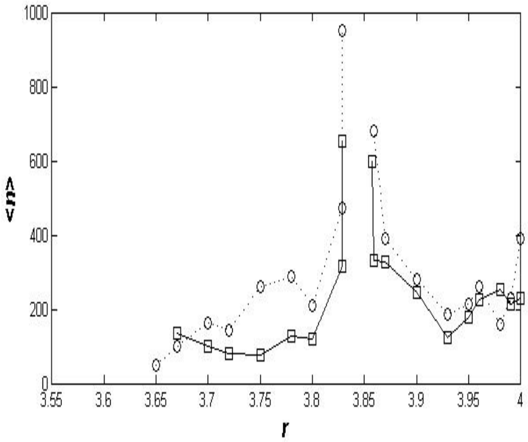

Fig.9. Mean recurrence time (<n>) for the cells [0.296875, 0.303125] and [0.696875, 0.703125], as a function of the control parameter value r , $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5r recurrence times. The initial condition for the first cell is $x_0= 0.3$ and for the second cell is $x_0= 0.7$. The straight line stands for the second cell and the dashed line is for the first cell .

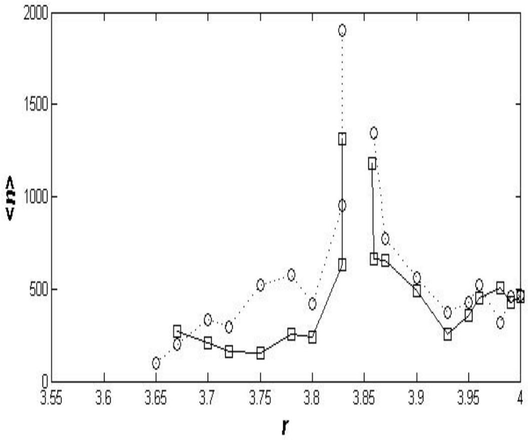

Fig.10. Mean recurrence time (<n>) for the cells [0.2984375, 0.3015625] and [0.6984375, 0.7015625], serve as a function of the control parameter value r, $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell I is $x_0 = 0.3$ and for the second cell is $x_0= 0.7$. The straight line stands for the second cell and the dashed line is for the first cell.

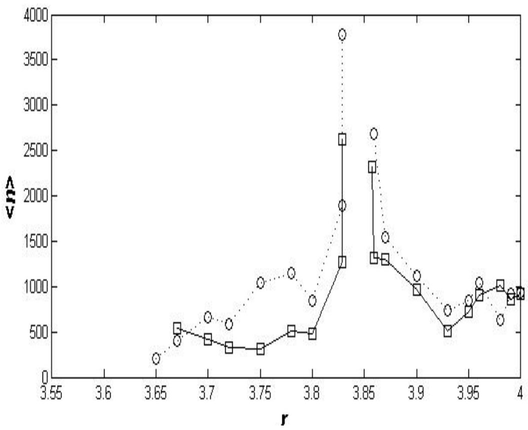

Fig.11. Mean recurrence time (<n>) for the cells [0.29921875, 0.30078125] and [0.69921875, 0.70078125], serve as a function of the control parameter value r, $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first2.10^5 recurrence times. The initial condition for the first cell is $x_0 = 0.3$ and for the second cell is $x_0= 0.7$. The straight line stands for the second cell and the dashed line is for the first cell.

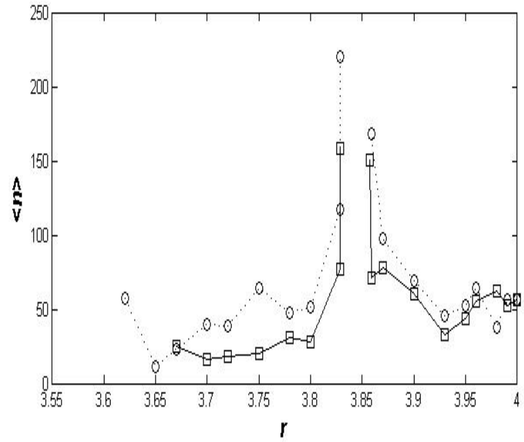

Fig.12. Mean recurrence time (<n>) for the cells [0.2875, 0.3125] and [0.6875, 0.7125], as a function of the control parameter value r , $3.55 < r < 4.00$ of the logistic map. We find the mean recurrence time by integrating over the first 2.10^5recurrence times. The initial condition for the first cell is $x_0 = 0.3$ and for the second cell is $x_0 = 0.7$. The straight line stands for the second cell and the dashed line is for the first cell.

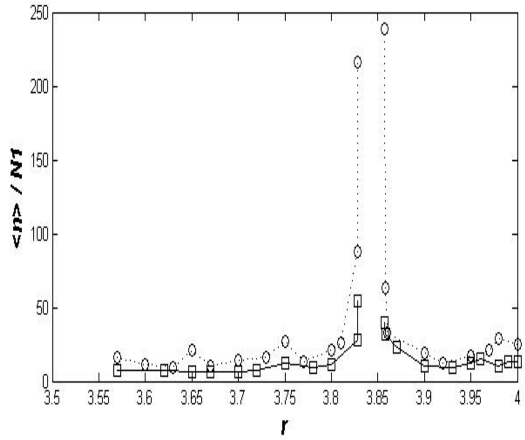

Fig.13.Comparison of the maximum first return time N_1 to the mean recurrence time (<n>) as a function of the control parameter value r, $3.55 < r < 4.00$ for the cell [0.25 , 0.35] .The straight line stands for the mean recurrence time and the dashed line is for the maximum first return time . There is a close accordance of the two curves.

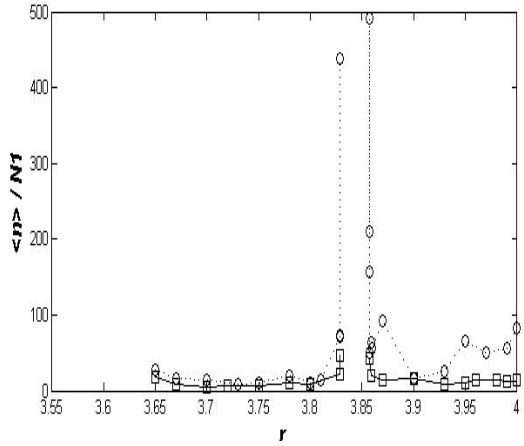

Fig.14.Comparison of the maximum first return time N_1 to the mean recurrence time (<n>) as a function of the control parameter value r, $3.55 < r < 4.00$ for the cell [0.65 , 0.75] .The straight line stands for the mean recurrence time and the dashed line is for the maximum first return time . There is a close accordance of the two curves

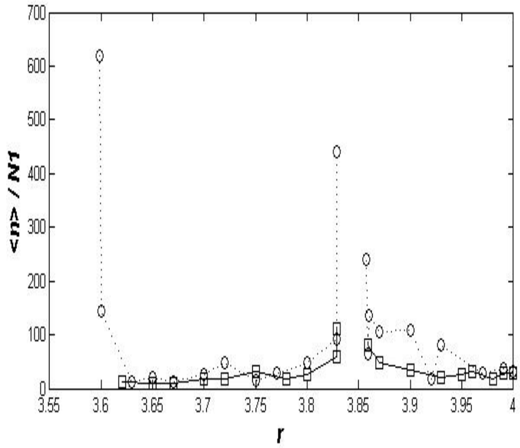

Fig.15.Comparison of the maximum first return time N_1 to the mean recurrence time (<n>) as a function of the control parameter value r, 3.55 < r < 4.00 for the cell [0.275 , 0.325].The straight line stands for the mean recurrence time and the dashed line is for the maximum first return time . There is a close accordance of the two curves

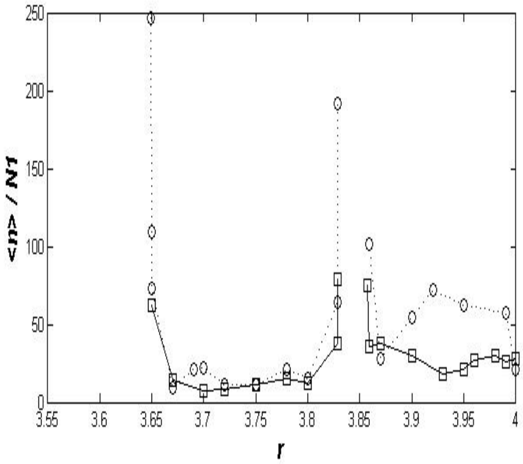

Fig.16.Comparison of the maximum first return time $N_1$to the mean recurrence time (<n>) as a function of the control parameter value r, 3.55 < r < 4.00 for the cell [0.675 , 0.725].The straight line stands for the mean recurrence time and the dashed line is for the maximum first return time . There is a close accordance of the two curves.

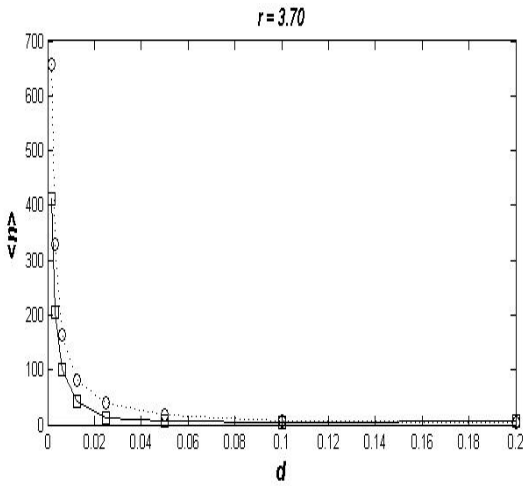

Fig.17. Mean recurrence (<n>) as a function of the diameter of the cell for different cell sizes d_i ranging from 0.0015625 to 0.2 . We remark that for $d_i \rightarrow 0$ the mean recurrence time becomes astronomically large .The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.The control parameter value r=3.70.

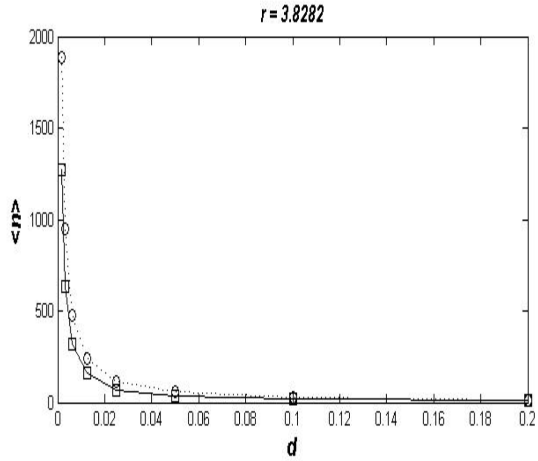

Fig.18. Mean recurrence (<n>) serves as a function of the diameter of the cell for different cell sizes d_i, ranging from 0.0015625 to 0.2. We remark that for $d_i \rightarrow 0$ the mean recurrence time becomes astronomically large .The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3. The control parameter value r=3.8282.

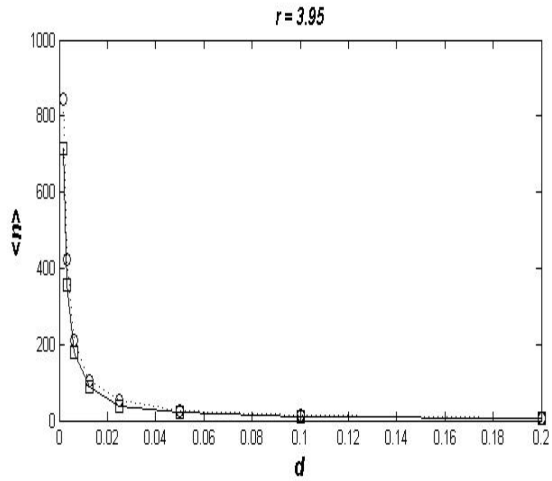

Fig.19. Mean recurrence (<n>) as a function of the diameter of the cell for different cell sizes d_i ranging from 0.0015625 to 0.2 . We remark that for $d_i \rightarrow 0$ the mean recurrence time becomes astronomically large .The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.The control parameter value r=3.95.

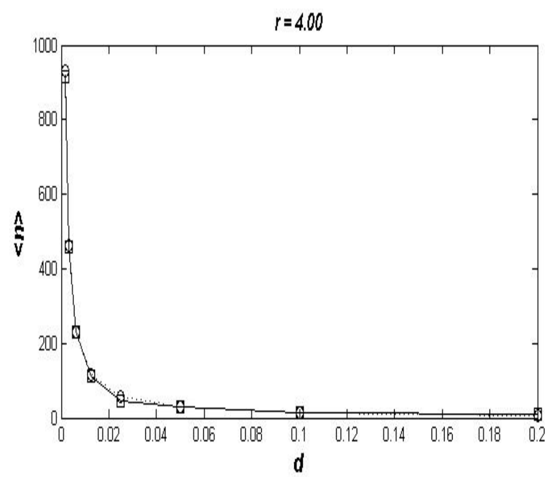

Fig.20. Mean recurrence (<n>) serves as a function of the diameter of the cell for different cell sizes d_i, ranging from 0.0015625 to 0.2. We remark that for $d_i \rightarrow 0$ the mean recurrence time becomes astronomically large .The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.The control parameter value r=4.00.

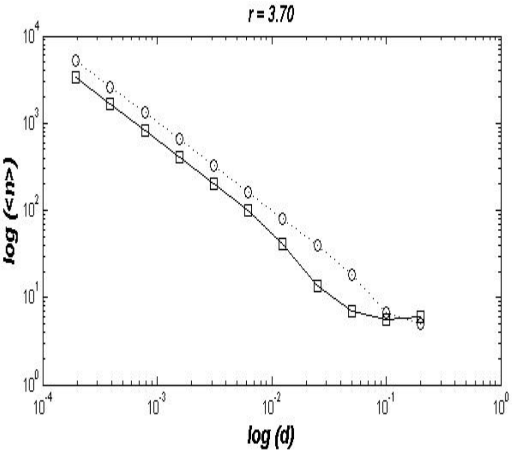

Fig.21: Log-Log representation of figure 17. We observe a linearity when d > 0.025.

The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.

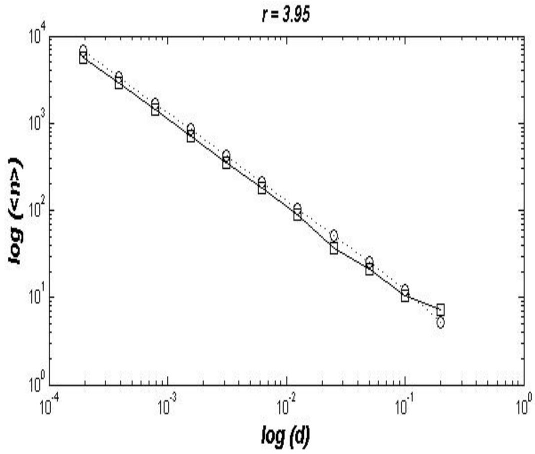

Fig.23. Log-Log representation of figure 19 .We observe a linearity when d > 0.025.

The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.

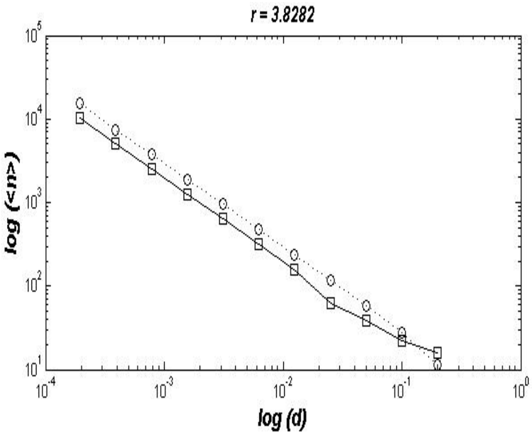

Fig.22. Log-Log representation of figure 18 .We observe a linearity when d > 0.025.

The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.

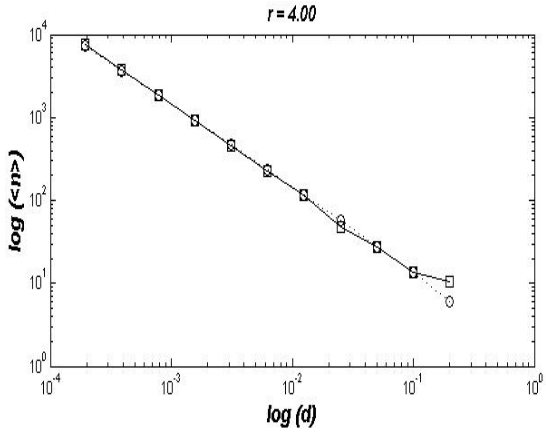

Fig.24. Log-Log representation of figure 20 .We observe a linearity when d > 0.025.

The center of the cells with the straight line is 0.7 and the center of the cells represented with the dashed line is 0.3.

TABL1. FIRST RETURN TIME (N_{FIRST}) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R. THE INITIAL CONDITION IS X_0= 0.3

d_i	cell	r=3.70	r=3.75	r=3.8282	r=3.8284	r=3.90	r=3.95	r=4.00
0.2	0.2-0.4	5	5	3	3	15	4	5
0.1	0.25-0.35	5	5	3	3	15	6	5
0.05	0.275-0.325	27	5	3	3	22	36	5
0.025	0.2875-0.3125	27	20	3	3	22	36	126
0.0125	0.29375-0.30625	27	20	3	3	22	98	126
0.00625	0.296875-0.303125	27	109	822	349	22	375	126

TABLE 2.FIRST RETURN TIME (N_{FIRST}) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS, CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R. THE INITIAL CONDITION IS X_0= 0.7

d_i	cell	r=3.70	r=3.75	r=3.8282	r=3.8284	r=3.90	r=3.95	r=4.00
0.2	0.6-0.8	4	8	1	1	13	15	21
0.1	0.65-0.75	6	10	12	12	16	19	21
0.05	0.675-0.725	8	10	25	12	18	19	21
0.025	0.6875-0.7125	10	10	69	12	18	60	42
0.0125	0.69375-0.70625	10	10	69	288	173	60	42
0.00625	0.696875-0.703125	10	10	69	375	487	60	342

TABLE3. MAXIMUM FIRST RETURN TIME (N_1) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS, CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE STARTING POINTS BELONG TO THE PARTITIONS. IF THE NUMBER OF ITERATIONS BYPASSES AN N_{MAX}=300,000.

Cell	**r=3.70**			**r=3.75**			**r=3.80**			**r=3.8282**		
	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}
0.2-0.4	0.27	14	0.2935	0.27	9	0.2987	0.27	10	0.2431	0.33	72	0.2207
0.25-0.35	0.27	14	0.2936	0.33	27	0.2689	0.29	21	0.2621	0.32	88	0.3053
0.275-0.325	0.30	27	0.3023	0.32	15	0.3199	0.279	49	0.300	0.277	93	0.2924
Cell	**r=3.8284**			**r=3.90**			**r=3.95**			**r=4.00**		
	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}
0.2-0.4	0.33	213	0.2018	0.27	18	0..3383	0.27	6	0.2024	0.25	inf	-
0.25-0.35	0.33	216	0.3293	0.33	19	0.2536	0.29	17	0.3184	0.31	25	0.3463
0.275-0.325	0.277	441	0.2897	0.279	109	0.3197	0.32	38	0.3100	0.31	31	0.2925

TABLE 4.MAXIMUM FIRST RETURN (N1)TIME AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS, CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE STARTING POINTS BELONG TO THE PARTITIONS. IF THE NUMBER OF ITERATIONS BY PASSES AN N_{MAX}=300,000.

Cell	**r=3.70**			**r=3.75**			**r=3.80**			**r=3.8282**		
	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}
0.6-0.8	0.75	8	0.7513	0.62	12	0.7797	0.79	11	0.7442	0.79	77	0.6515
0.65-0.75	0.69	15	0.6514	0.70	10	0.7005	0.70	11	0.6842	0.67	73	0.6585
0.675-0.725	0.679	22	0.7249	0.71	11	0.7055	0.677	15	0.6752	0.71	64	0.6969
Cell	**r=3.8284**			**r=3.90**			**r=3.95**			**r=4.00**		
	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}	X_0	N	X_{return}
0.6-0.8	0.73	218	0.6670	0.73	38	0.6918	0.72	36	0.6423	0.77	35	0.7974
0.65-0.75	0.67	438	0.6952	0.66	17	0.6942	0.66	66	0.6978	0.69	82	0.6657
0.675-0.725	0.72	192	0.7219	0.72	54	06994	0.72	62	0.7105	0.70	21	0.6848

TABLE 5.MEAN RECURRENCE TIME (<N>) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE INITIAL CONDITION IS X0= 0.3 AND AN UPPER BOUND OF ITERATIONS IS 107. THE STATISTICAL MEAN VALUE IS TAKEN OVER THE FIRST 200,000 RECURRENCE TIMES.

d_i	**cell**	**r=3.70**	**r=3.75**	**r=3.80**	**r=3.8282**	**r=3.8284**	**r=3.90**	**r=3.95**	**r=3.98**	**r=4.00**
0.2	0.2-0.4	5.0775	4.2021	4.6894	11.5420	23.8505	4.3657	5.1314	5.1835	6.1081
0.1	0.25-0.35	6.6658	12.6683	11.1305	27.8682	54.6147	10.4127	12.0976	10.5381	13.3368
0.05	0.275-0.325	18.5411	31.2349	24.9299	57.9592	110.5198	33.8717	25.0850	18.8737	27.7296
0.025	0.2875-0.3125	39.8885	64.1448	51.5921	117.2870	220.0743	68.9780	52.0628	37.7888	56.7182
0.0125	0.29375-0.30625	81.2343	129.8509	104.3507	236.6347	474.6542	138.9726	105.3790	79.1376	114.2374
0.00625	0.296875-0.303125	164.5462	260.8450	209.5391	472.8957	949.1042	278.8684	211.8284	160.3389	230.3852
0.003125	0.2984375-0.3015625	329.5387	520.4983	419.2364	949.9452	1905.5	558.6443	423.6301	319.7258	464.3290
0.0015625	0.29921875-0.30078125	658.5070	1045.5	840.7165	1888.8	3786.2	1124.8	844.9981	638.5226	932.4152
0.00078125	0.299609375-0.300390625	1314.2	2092.8	1694.8	3728.5	7524.6	2236.3	1687.1	1277.6	1828.4
0.000390625	0.2998046875-0.3001953125	2620.9	4194.5	3392.7	7451.9	15414.0	4440.0	3394.3	2552.2	3641.5
0.0001953125	0.29990234375-0.30009765625	5215.4	8255.2	6820.5	15181.0	31102	8906.9	6823.7	5050.2	7275.4

TABLE6. MEAN RECURRENCE TIME (<N>) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE INITIAL CONDITION IS X0= 0.7 AND AN UPPER BOUND OF ITERATIONS IS 107.. THE STATISTICAL MEAN VALUE IS TAKEN OVER THE FIRST 200,000 RECURRENCE TIMES.

d_i	**cell**	**r=3.70**	**r=3.75**	**r=3.80**	**r=3.8282**	**r=3.8284**	**r=3.90**	**r=3.95**	**r=3.98**	**r=4.00**
0.2	0.6-0.8	5.9319	4.9148	5.9957	15.6456	33.3450	11.9549	7.2500	10.0527	10.3164
0.1	0.65-0.75	5.4889	6.7170	7.7874	21.7837	46.6712	15.7662	10.6231	14.3933	13.3399
0.05	0.675-0.725	7.0667	11.3407	12.7084	37.9664	78.9091	29.4631	21.3674	29.7274	27.7692
0.025	0.6875-0.7125	13.6339	17.1727	22.5358	63.5620	131.7063	49.9453	36.3370	21.9338	47.0023
0.0125	0.69375-0.70625	41.5967	34.4578	59.5738	157.8750	320.2815	121.0868	88.3488	125.9337	113.9201
0.00625	0.696875-0.703125	100.8996	76.6864	119.7173	316.5723	651.0475	243.8168	177.8460	252.8347	228.8377
0.003125	0.6984375-0.7015625	205.2227	154.7390	239.5198	635.2441	1314.2	490.0061	357.4575	506.0038	458.8979
0.0015625	0.69921875-0.70078125	411.9750	316.8357	477.6817	1269.1	2617.3	972.3010	715.3497	1011.6	916.0717
0.00078125	0.699609375-0.700390625	823.6489	644.6494	950.1185	2537.5	5292.4	2023.2	1431.3	2026.0	1848.0
0.000390625	0.6998046875-0.7001953125	1659.9	1240.9	1887.0	5088.8	10476	4086.2	2845.1	3966.6	3777.0
0.0001953125	0.69990234375-0.70009765625	3310.0	2563.7	3689.3	10171	20910.0	8160.1	5676.0	7927.8	7671.2

TABLE7. MEAN RECURRENCE TIME (<N>) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE INITIAL CONDITION IS X0= 0.7 AND THE UPPER BOUND OF ITERATIONS IS 107.THE STATISTICAL MEAN VALUE IS TAKEN OVER THE FIRST 500,000 RECURRENCE TIMES (AUGMENTED STATISTICS).

d_i	**cell**	**r=3.70**	**r=3.8282**	**r=3.8284**	**r=3.90**	**r=3.95**	**r=4.00**
0.2	0.6-0.8	5.9294	15.6381	33.3214	11.9589	7.2450	10.2858
0.1	0.65-0.75	5.4874	21.8499	46.6805	15.7944	10.6084	13.3296
0.05	0.675-0.725	7.0766	37.9146	78.6101	29.4935	21.3361	27.7666
0.025	0.6875-0.7125	15.8993	76.7253	158.0788	59.965	43.7202	56.4723
0.0125	0.69375-0.70625	41.5321	158.0612	320.0213	120.8734	88.5441	114.0342
0.00625	0.696875-0.703125	100.5765	316.8964	651.1761	244.1396	178.1602	229.3062
0.003125	0.6984375-0.7015625	205.2435	634.8872	1314.4	490.3106	357.3884	458.7276

TABLE8. MEAN RECURRENCE TIME (<N>) AS A FUNCTION OF DIAMETER OF THE DIFFERENT CELLS CONSIDERED HERE FOR DIFFERENT CONTROL PARAMETER VALUES R .THE INITIAL CONDITION IS X_0= 0.3 AND THE UPPER BOUND OF ITERATIONS IS 10^7. THE STATISTICAL MEAN VALUE IS TAKEN OVER THE FIRST 500,000 RECURRENCE TIMES (AUGMENTED STATISTICS) .

d_i	**cell**	**r=3.70**	**r=3.8282**	**r=3.8284**	**r=3.90**	**r=3.95**	**r=4.00**
0.2	0.2-0.4	5.0775	11.5317	23.7934	4.3605	5.1319	6.1044
0.1	0.25-0.35	6.6683	27.7956	54.5870	10.4128	12.1193	13.3256
0.05	0.275-0.325	18.5403	58.0306	110.3981	33.9144	24.9756	27.8031
0.025	0.2875-0.3125	39.8787	117.3031	219.9903	68.8125	52.0964	56.7359
0.0125	0.29375-0.30625	81.5729	236.2184	474.5695	138.9452	105.2671	114.1946
0.00625	0.296875-0.303125	164.3744	472.5079	948.9736	279.2894	211.5931	230.3550
0.003125	0.2984375-0.3015625	329.1030	949.4720	1905.1	560.0583	423.8352	464.2334

TABLE 9 APPLICATION OF THE LEAST SQUARE METHOD TO THE LOG – LOG PLOT OF THE MEAN RECURRENCE TIME AS A FUNCTION OF THE DIAMETER OF THE CELLS WHERE THE CENTER OF THE CELLS IS 0.3 . WE OBTAIN BEST FIT FOR THE LINEAR REGRESSION *LOG (<N>) = -A LOG (D) + C.*

	r=3.70	**r=3.8282**	**r=3.95**	**r=4.00**
a	-1.0032	-0.9978	-1.0031	-1.0154
c	0.0015	1.0945	0.2588	0.2577
error	6.1405×10^{-4}	3.3765×10^{-4}	1.3495×10^{-4}	0.0136

TABLE. 10 APPLICATION OF THE LEAST SQUARE METHOD TO THE LOG – LOG PLOT OF THE MEAN RECURRENCE TIME AS A FUNCTION OF THE DIAMETER OF THE CELLS WHERE THE CENTER OF THE CELLS IS 0.7. WE OBTAIN BEST FIT FOR THE LINEAR REGRESSION LOG (<N>) = -A LOG (D) + C.

	r=3.70	**r=3.8282**	**r=3.95**	**r=4.00**
a	-1.0065	-1.0208	-1.0006	-1.0114
c	-0.4867	0.5349	0.1027	0.2964
error	1.2751×10^{-4}	0.0306	1.2746×10^{-4}	4.3249×10^{-4}

VI. CONCLUSIONS AND DISCUSSION

The problem of recurrence times of the logistic map is discussed in detail. After introducing formulation as well as useful definitions, the problem is cast in a rigorous plan. We introduce numerical experimentation and we measure first return times, maximum first return times and mean recurrence times, for different partitions , and a broad spectrum of candidate behaviors of the logistic map .

Concerning first return times, there are similarities in the behaviors of different partitions. As the partition becomes very small and finally is shrunk to a point $(d_i \to 0)$ there is a stabilization of the first return time. Also, measurable statistical fluctuations are manifested going from one partition to the other partition .The above properties are in very good agreement with the theoretical predictions of statistical physics.

For the maximum first return time, extended numerical computations show that they depend strongly on the initial point of the trajectory and the control parameter value of the map. As far as we know these results are original.

In the third class of our numerical experiments we have studied in some detail the mean recurrence times. We have shown that the mean recurrence is well defined, and connected with the very notion of ergodicity. Statistical trends are present showing that for the partitions of an important diameter fluctuations are small, and a thermodynamic behavior is rigorously established. For partitions of a very small diameter the fluctuations are ubiquitous, and in this limit $(d_i \to 0)$ the mean recurrence time is inversely proportional to the diameter of the cell.

In the opinion of the authors the results exposed here have also a profound pedagogical value. More specifically, the study of mechanical behavior of matter, with its periodicities and instabilities are better understood under the prism of study of the logistic map. The ergodic theory is well reflected in the statistical behavior of recurrence and mean recurrence times. The one-dimensional maps are an ideal Laboratory for the in-depth study of Statistical Mechanics.

We take the opportunity to discuss actual and further connections with information technology and experimental mathematics .In particular information technology obeying Moore's law allows to everybody to perform careful numerical experiments for the study of statistical mechanics. Moreover the computer much like as a telescope sheds light in the branch of science qualified as experimental mathematics ([8]-[10]) .

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